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Several solution methods for the generalized complex eigenvalue problem with bounded uncertainties

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Abstract

The aim of this paper is to evaluate the effects of uncertain-but-bounded parameters on the complex eigenvalues of the non-proportional damping structures. By combining the interval mathematics and the finite element analysis, the mass matrix, the damping matrix and the stiffness matrix were represented as the interval matrices. Firstly, with the help of the optimization theory, we presented an exact solution—the vertex solution theorem, for determining the exact upper bounds or maximum values and exact lower bounds or minimum values of complex eigenvalues of structures, where the extreme values are reached on the boundary of the interval mass, damping and stiffness matrices. Then, an interval perturbation method was proposed, which needs less computational efforts. A numerical example of a seven degree-of-freedom spring-damping-mass system was used to illustrate the computational aspects of the presented vertex solution theorem and the interval perturbation method in comparison with Deif's method.

Keywords: Interval complex eigenvalue; Non-proportional damping; Vertex solution theorem; Interval perturbation method; Deif's method

1. Introduction

Complex eigenvalues estimation of non-proportional damping structures with uncertainty is very vital to the design and analysis of structures used in many engineering problems. All structural analysis and design problems involve imprecision or approximation or uncertainty. Analysis and design under uncertainty depend on representation of what is known about the uncertain information. The choice of a model of

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uncertainty depends on the type and quantity of information available. There are three classes of uncertain models (Li and Liao, 2001), and they are: probabilistic approach, fuzzy theory and interval analysis. If the uncertain variables are described as random variables or random process with a specified probability distribution, the probabilistic approach can be used. In fuzzy theory, the uncertainty is interpreted as the designer and analyst's choice to use a particular value for the uncertain variable, if a preference function is used to describe the desirability of using different values within the some range. In interval analysis, the uncertain variables are denoted by a simple range or set, i.e. interval vector. It can be seen that when information about uncertain variables in the form of a preference or probability function is not available, interval analysis can be used most conveniently.

In many dynamic problems of structural engineering, one often encounters the following problem: Given two real symmetric non-positive definite matrices A and B of order n, determine the value of a scalar λ which satisfies the equation $Au = \lambda Bu$, which is called the generalized complex eigenvalue problem. λ is called the eigenvalue of matrix pair A and B, and u is the associated eigenvector. The eigenvalue λ is generally to be a complex number $\lambda = \lambda_r + \sqrt{-1}\lambda_y$, where λ_r and λ_y are respectively the real and imaginary parts of the complex eigenvalue λ for identical structural system. However, experiments have shown that complex eigenvalue varies in uncertainty because the elements of the matrices A and B can be neither measured exactly nor calculated exactly. The incomplete information about the elements of the matrices A and B is a result of errors in observation, measurement errors, change on operating conditions ageing, maintenance-induced errors, etc. In such circumstance we do not know the elements of the matrices A and B exactly; instead, we only know the ends of intervals in which the elements of the matrices A and B are confined. Thus, the complex eigenvalues are uncertain variables whose uncertain properties are determined by the uncertain elements of the matrices A and B.

When the matrices A and B are interval uncertain (Moore, 1979; Alefeld and Herzberger, 1983; Deif, 1991), i.e., $A^{\mathrm{I}} = [\underline{A}, \overline{A}] = (a_{ij}^{\mathrm{I}}) = ([\underline{a}_{ij}, \bar{a}_{ij}])$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}] = (b_{ij}^{\mathrm{I}}) = ([\underline{b}_{ij}, \bar{b}_{ij}])$, the complex eigenvalues $\lambda = \lambda_r + \sqrt{-1}\lambda_y$ will be interval complex numbers $\lambda_i^{\mathrm{I}} = \lambda_{ir}^{\mathrm{I}} + \sqrt{-1}\lambda_{iy}^{\mathrm{I}}$, where $\lambda_{ir}^{\mathrm{I}} = [\underline{\lambda}_{ir}, \bar{\lambda}_{ir}]$, $\lambda_{iy}^{\mathrm{I}} = [\underline{\lambda}_{iy}, \bar{\lambda}_{iy}]$, $i = 1, 2, \ldots, n$.

In this study, we will deal with the generalized interval complex eigenvalue problem of real symmetric but non-positive definite interval matrices.

The research on the interval eigenvalue problem has begun to emerge in recent years. Extensive researches were investigated by control engineers (Bialas, 1983; James, 1984) from the view point of stability and robustness analysis. Hudak (1984) investigated ways of finding a constant matrix such that the certain constraint condition with the interval matrix given. Rohn (1987) studied the symmetric interval matrix and ended up with the formulae when the wide of interval matrix has rank one. Hollot and Bartlett (1987) verified that the spectrum of eigenvalues of an interval matrix family was found to depend of finding a constant matrix under the constraint with the given interval matrix. Based on the invariance properties of the characteristic vectors' entries, Deif (1991) obtained the solution theorem for interval matrix. Qiu et al. (1995) extended Deif's solution theorem for the standard interval eigenvalue problem of real symmetric interval matrices to the generalized interval eigenvalue problem, and presented the interval perturbation method, semi-definite solution theorem and the inclusion theorem.

However, the mentioned-above methods are all used to solve the standard or generalized real eigenvalue problem. Deif (1991) established the solution theorem for standard and generalized interval complex eigenvalue problem; nevertheless, there existed many difficulties, such as: it is quite difficult how to determine the invariance properties of the eigenvectors' components in the interval matrix; large computational efforts.

In this paper, firstly, by virtue of interval mathematics and the optimization theory, an exact solution method—the vertex solution theorem, was proposed for determining the exact maximum values or the upper bounds and the minimum values or the lower bounds on the generalized interval complex eigenvalues of structures with uncertain-but-bounded parameters; then an efficient interval perturbation method was presented.

In the following, first the problem formulation of the generalized interval complex eigenvalue of non-proportional damping structure with uncertain-but-bounded parameters is given in Section 2. Then in Section 3 the vertex solution theorem for the generalized interval complex eigenvalue problem is presented, followed by the interval perturbation method in Section 4. An example of a seven degree-of-freedom spring-damping-mass system is used to illustrate the application of the presented methods in Section 5. Finally, in Section 6 a conclusion is given.

2. Problem formulation

Let us consider the differential equation of motion of systems with n degrees of freedom (Müeller and Schiehlen, 1985)

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) = Q(t) \tag{1}$$

where $M = (m_{ij})$ is the mass matrix; $C = (c_{ij})$ is the damping matrix; $K = (k_{ij})$ is the stiffness matrix, and $Q(t) = (q_i(t))$ is the external load vector. The matrices M, C and K are all symmetric, and are $n \times n$-dimensional matrices. $y(t) = (y_i(t)),\ \dot{y}(t) = (\dot{y}_i(t))$ and $\ddot{y}(t) = (\ddot{y}_i(t))$ are the n-dimensional displacement, velocity, and acceleration vectors.

The dynamics problem associated with Eq. (1) can be reduced to a standard form by a method developed by Meirovotch (1980). In the following we will give a brief description of the method.

Introducing the $2n$-dimensional state vector

$$x(t) = (y(t)^{\mathrm{T}}\dot{y}(t)^{\mathrm{T}})^{\mathrm{T}} \tag{2}$$

and the $2n$-dimensional excitation vector

$$F(t) = (Q(t)^{\mathrm{T}}0^{\mathrm{T}})^{\mathrm{T}} \tag{3}$$

Eq. (1) can be written in the form

$$A\dot{x}(t) + Bx(t) = F(t) \tag{4}$$

where

$$A = \begin{pmatrix} C & M \\ M & 0 \end{pmatrix}, \quad B = \begin{pmatrix} K & 0 \\ 0 & -M \end{pmatrix} \tag{5}$$

in which A and B are $2n \times 2n$-dimensional real symmetric non-positive definite matrices.

The generalized complex eigenvalue problem corresponding to Eq. (4) has the following form:

$$Ax = \lambda Bx \tag{6}$$

Obviously, the solutions to Eq. (6) consist of $2n$ complex eigenvalues $\lambda_i = \lambda_{ri} + \sqrt{-1}\lambda_{yi}$ and $2n$ complex eigenvectors $x_i = x_{ri} + \sqrt{-1}x_{yi}$. Because the matrices A and B are real symmetric, if $\lambda_i = \lambda_{ri} + \sqrt{-1}\lambda_{yi}$ is an eigenvalue, then its complex conjugate $\tilde{\lambda}_i = \lambda_{ri} - \sqrt{-1}\lambda_{yi}$ is also an eigenvalue, and a similar statement can be made concerning the eigenvectors.

The complex eigenvalues given by Eq. (6) are often assumed to be constant for identical structural systems. However, experience and experiments have shown that these values vary uncertainly because in reality the physical and geometric properties of the elements in A and B can be neither measured exactly nor manufactured exactly. In this paper, we assume that the uncertainties in A and B are bounded, and the uncertain but bounded matrices A and B can be written as the following matrix inequality form

$$\underline{A} \leqslant A \leqslant \overline{A}, \quad \underline{B} \leqslant B \leqslant \overline{B} \tag{7a}$$

or the element form

$$\underline{a}_{ij} \leqslant a_{ij} \leqslant \bar{a}_{ij}, \quad \underline{b}_{ij} \leqslant b_{ij} \leqslant \bar{b}_{ij}, \ \ i,j = 1,2,\ldots,n \tag{7b}$$

in which $\overline{A} = (\bar{a}_{ij})$ and $\underline{A} = (\underline{a}_{ij})$, respectively, are the upper bound matrix and the lower bound matrix of the uncertain matrix A, and $\overline{B} = (\bar{b}_{ij})$ and $\underline{B} = (\underline{b}_{ij})$, respectively, are the upper bound matrix and the lower bound matrix of the uncertain matrix B.

Usually, it is very difficult to solve the generalized complex eigenvalue problem Eq. (6) under the condition of the matrix inequality constraints Eqs. (7).

We number the real parts and imaginary parts of the complex eigenvalues of the matrix pair $\langle A, B\rangle$ in non-descending order

$$\lambda_{r1} \leqslant \lambda_{r2} \leqslant \cdots \leqslant \lambda_{r2n}, \quad \lambda_{y1} \leqslant \lambda_{y2} \leqslant \cdots \leqslant \lambda_{y2n} \tag{8}$$

In this paper, we shall study a method for computing the complex eigenvalues of Eq. (6) subjected to the constraints Eqs. (7), in which the elements a_{ij}, and b_{ij}, $i,j = 1,2,\ldots,2n$, of the matrix pair A and B are not known precisely. The incomplete information about the elements of matrices A and B is a result of measurement errors, etc. In most cases we only know the ends of intervals in which the elements of the matrices A and B are confined.

In terms of the interval matrix notation (Moore, 1979; Alefeld and Herzberger, 1983), the constraint condition Eq. (7a) can be expressed as

$$A \in A^{\mathrm{I}}, \quad B \in B^{\mathrm{I}} \tag{9a}$$

and

$$a_{ij} \in a_{ij}^{\mathrm{I}} = [\underline{a}_{ij}, \bar{a}_{ij}], \quad b_{ij} \in b_{ij}^{\mathrm{I}} = [\underline{b}_{ij}, \bar{b}_{ij}], \ \ i,j = 1,2,\ldots,2n \tag{9b}$$

in which $A^{\mathrm{I}} = [\underline{A}, \overline{A}]$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}]$ are the symmetric interval matrices.

There are many applications in Eq. (6) with the constraint conditions Eqs. (7), where the elements of the matrices A and B are not precisely known. If we known an interval matrix A^{I} that is bounding A and an interval matrix B^{I} that is bounding B, the generalized complex eigenvalue problem can be expressed as

$$A^{\mathrm{I}}x = \lambda B^{\mathrm{I}}x \tag{10}$$

which is called the generalized interval complex eigenvalue problem. Because A^{I} and B^{I} are defined as interval matrices, the associated real parts and imaginary parts of their complex eigenvalues similarly constitute interval variables

$$\lambda^{\mathrm{I}} = \lambda_r^{\mathrm{I}} + \sqrt{-1}\lambda_y^{\mathrm{I}} = [\underline{\lambda}_r, \bar{\lambda}_r] + \sqrt{-1}[\underline{\lambda}_y, \bar{\lambda}_y] = (\lambda_i^{\mathrm{I}}) = (\lambda_{ri}^{\mathrm{I}} + \sqrt{-1}\lambda_{yi}^{\mathrm{I}}) \tag{11}$$

where

$$\lambda_{ri}^{\mathrm{I}} = [\underline{\lambda}_{ri}, \bar{\lambda}_{ri}, \quad \lambda_{yi}^{\mathrm{I}} = [\underline{\lambda}_{yi}, \bar{\lambda}_{yi}], \ \ i,j = 1,2,\ldots,2n \tag{12}$$

The interval eigenvalue solutions to Eq. (10) will contain the eigenvalue solutions to Eqs. (6) and (7). The real parts and the imaginary parts of the complex eigenvalue solutions to Eqs. (6) and (7) are denoted by the following sets:

$$\Gamma_r = \{\lambda_r : \lambda_r \in R^n, \ Ax = (\lambda_r + \sqrt{-1}\lambda_y)Bx, \ A = A^{\mathrm{T}}, \ B = B^{\mathrm{T}}, \ A \in A^{\mathrm{I}}, \ B \in B^{\mathrm{I}}\} \tag{13}$$

and

$$\Gamma_y = \left\{\lambda_y : \lambda_y \in R^n, \ Ax = (\lambda_r + \sqrt{-1}\lambda_y)Bx, \ A = A^{\mathrm{T}}, \ B = B^{\mathrm{T}}, \ A \in A^{\mathrm{I}}, \ B \in B^{\mathrm{I}}\right\} \tag{14}$$

We should stress that Γ_r and Γ_y may be generally of complicated geometric shapes so that it may usually impractical to try to solve them. Instead, it is a common practice to seek the interval vector of the real part $\lambda_r^{\mathrm{I}} = [\underline{\lambda}_r, \overline{\lambda}_r] = (\lambda_{ri}^{\mathrm{I}})$ containing Γ_r and the interval vector of the imaginary part $\lambda_y^{\mathrm{I}} = [\underline{\lambda}_y, \overline{\lambda}_y] = (\lambda_{yi}^{\mathrm{I}})$ containing Γ_y, where the interval vectors λ_r^{I} and λ_y^{I} have the narrowest possible interval components.

When endeavoring to solve Eq. (10), we are, in fact, able to determine an interval vector of the real part $\lambda_r^{\mathrm{I}} = [\underline{\lambda}_r, \overline{\lambda}_r] = (\lambda_{ri}^{\mathrm{I}})$ and an interval vector of the imaginary part $\lambda_y^{\mathrm{I}} = [\underline{\lambda}_y, \overline{\lambda}_y] = (\lambda_{yi}^{\mathrm{I}})$, which all have the smallest widths of intervals enclosing all possible complex eigenvalues $\lambda \in C^{2n}$, satisfying $Ax = \lambda Bx$, when A and B assume all possible combinations inside A^{I} and B^{I}. In other words, we seek a multi-dimensional rectangle $\lambda^{\mathrm{I}} = \lambda_r^{\mathrm{I}} + \sqrt{-1}\lambda_y^{\mathrm{I}} = [\underline{\lambda}_r, \overline{\lambda}_r] + \sqrt{-1}[\underline{\lambda}_y, \overline{\lambda}_y] = (\lambda_i^{\mathrm{I}}) = (\lambda_{ri}^{\mathrm{I}} + \sqrt{-1}\lambda_{yi}^{\mathrm{I}})$ containing all eigenvalues of Eqs. (6) and (7).

In this study, we shall present a solution of the generalized interval complex eigenvalue problem Eq. (10) which serves a wide range of applications. The basic problem to be solved herein as follows: Given the central matrices $A^{\mathrm{c}} = (\overline{A} + \underline{A})/2$ and $B^{\mathrm{c}} = (\overline{B} + \underline{B})/2$ of $A^{\mathrm{I}} = [\underline{A}, \overline{A}]$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}]$, respectively, and the deviation amplitude matrices $\Delta A = (\overline{A} - \underline{A})/2$ and $\Delta B = (\overline{B} - \underline{B})/2$ of $A^{\mathrm{I}} = [\underline{A}, \overline{A}]$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}]$, find a multi-dimensional rectangle containing the set of eigenvalues of Eq. (10) for interval matrices $A^{\mathrm{I}} = [\underline{A}, \overline{A}] = \{A : |A - A^{\mathrm{c}}| \leqslant \Delta A\}$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}] = \{B : |B - B^{\mathrm{c}}| \leqslant \Delta B\}$. In other words, we seek the interval complex eigenvalues or the upper and lower bounds on the real parts and the imaginary parts of the interval complex eigenvalues or the set of Eq. (10), i.e.

$$\lambda^{\mathrm{I}} = \lambda_r^{\mathrm{I}} + \sqrt{-1}\lambda_y^{\mathrm{I}} = [\underline{\lambda}_r, \overline{\lambda}_r] + \sqrt{-1}[\underline{\lambda}_y, \overline{\lambda}_y] = (\lambda_i^{\mathrm{I}}) = (\lambda_{ri}^{\mathrm{I}} + \sqrt{-1}\lambda_{yi}^{\mathrm{I}}) \tag{15}$$

where

$$\lambda_{ri}^{\mathrm{I}} = [\underline{\lambda}_{ri}, \overline{\lambda}_{ri}, \quad \lambda_{yi}^{\mathrm{I}} = [\underline{\lambda}_{yi}, \overline{\lambda}_{yi}], \;\; i,j = 1,2,\ldots,2n \tag{16}$$

in which

$$\overline{\lambda}_{ri} = \max_{A \in A^{\mathrm{I}}, B \in B^{\mathrm{I}}} \{\lambda_{ri}(\langle A, B \rangle)\}, \quad \underline{\lambda}_{ri} = \min_{A \in A^{\mathrm{I}}, B \in B^{\mathrm{I}}} \{\lambda_{ri}(\langle A, B \rangle)\}, \;\; i = 1,2,\ldots,2n \tag{17}$$

and

$$\overline{\lambda}_{yi} = \max_{A \in A^{\mathrm{I}}, B \in B^{\mathrm{I}}} \{\lambda_{ri}(\langle A, B \rangle)\}, \quad \underline{\lambda}_{yi} = \min_{A \in A^{\mathrm{I}}, B \in B^{\mathrm{I}}} \{\lambda_{ri}(\langle A, B \rangle)\}, \;\; i = 1,2,\ldots,2n \tag{18}$$

where

$$\lambda_{ri} = \lambda_{ri}(\langle A, B \rangle) = \min_{S_i \subset C^{2n}} \max_{\substack{x \neq 0 \\ x \in S_i}} \mathrm{Re}\left\{\frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}Bx}\right\}, \quad i = 1,2,\ldots,2n \tag{19}$$

and

$$\lambda_{yi} = \lambda_{yi}(\langle A, B \rangle) = \min_{S_i \subset C^{2n}} \max_{\substack{x \neq 0 \\ x \in S_i}} \mathrm{Im}\left\{\frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}Bx}\right\}, \quad i = 1,2,\ldots,2n \tag{20}$$

where $S_i \subset C^{2n}$ is an arbitrary i-dimensional sub-space (see Appendix A).

Obviously, the maximum problems and the minimum problems in Eqs. (17) and (18) are global optimization problems.

3. The vertex solution theorem

Before introducing the theorem that is the subject of this section, some notations are given. The boundary matrices or extreme point matrices or vertex matrices of the $2n \times 2n$-dimensional interval matrices $A^{\mathrm{I}} = [\underline{A}, \overline{A}]$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}]$ are, respectively, defined by

$$\hat{A}_l = \left\{ \hat{A}_l = (\hat{a}^l_{ij}) \in A^{\mathrm{I}} : \hat{a}^l_{ij} = \hat{a}^l_{ji} = \overline{a}_{ij} (\text{or } \underline{a}_{ij}), i,j = 1,2,\ldots,2n \right\} \quad l = 1,2,3,\ldots,2^{2n\times 2n} \tag{21}$$

and

$$\hat{B}_l = \left\{ \hat{B}_l = (\hat{b}^l_{ij}) \in B^{\mathrm{I}} : \hat{b}^l_{ij} = \hat{b}^l_{ji} = \overline{b}_{ij} (\text{or } \underline{b}_{ij}), i,j = 1,2,\ldots,2n \right\} \quad l = 1,2,3,\ldots,2^{2n\times 2n} \tag{22}$$

Under the matrix inequality constraint condition Eq. (7a), let us consider the minimax Rayleigh quotient of real part and the imaginary part of the matrix pair $\langle A, B \rangle$.

$$\lambda_{ri} = \lambda_{ri}(\langle A,B \rangle) = \min_{S_i \subset C^n} \max_{\substack{x \neq 0 \\ x \in S_i}} \mathrm{Re}\left\{ \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} B x} \right\}, \quad i = 1,2,\ldots,2n \tag{23}$$

and

$$\lambda_{yi} = \lambda_{yi}(\langle A,B \rangle) = \min_{S_i \subset C^n} \max_{\substack{x \neq 0 \\ x \in S_i}} \mathrm{Im}\left\{ \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} B x} \right\}, \quad i = 1,2,\ldots,2n \tag{24}$$

In the sequel, we only prove the vertex solution theorem for the real part of the complex eigenvalue, because the proof of the vertex solution theorem for the imaginary part of the complex eigenvalue is similar to that for the real part of the complex eigenvalue.

Obviously, according to the definition of the quadratic form, the extreme problem of the minimax Rayleigh quotient of the matrix pair $\langle A, B \rangle$ may also be written in the following useful form

$$\lambda_{ri} = \lambda_{ri}(\langle A,B \rangle) = \min_{S_i \subset C^n} \max_{\substack{x \neq 0 \\ x \in S_i}} \mathrm{Re}\left\{ \frac{\sum_{p,q=1}^{2n} a_{pq} x_p x_q}{\sum_{p,q=1}^{2n} b_{pq} x_p x_q} \right\}, \quad i = 1,2,\ldots,2n \tag{25}$$

subject to the element inequality constraint condition Eq. (7b).

The problem Eq. (25) subject to the inequalities Eq. (7b) can simply be written as the extreme value problem as follows:

$$\lambda_{ri\mathrm{ext}} = \lambda_{ri\mathrm{ext}}(\langle A,B \rangle) = \min_{S_i \subset C^n} \max_{\substack{x \neq 0 \\ x \in S_i}} \left\{ \operatorname*{extrem}_{\substack{a_{pq} \in a^{\mathrm{I}}_{pq}, b_{pq} \in b^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \left\{ \mathrm{Re}\left\{ \frac{\sum_{p,q=1}^{2n} a_{pq} x_p x_q}{\sum_{p,q=1}^{2n} b_{pq} x_p x_q} \right\} \right\} \right\}, \quad i = 1,2,\ldots,2n \tag{26}$$

For the extreme value problem

$$R = \operatorname*{extrem}_{\substack{a_{pq} \in a^{\mathrm{I}}_{pq}, b_{pq} \in b^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \left\{ \mathrm{Re}\left\{ \frac{\sum_{p,q=1}^{2n} a_{pq} x_p x_q}{\sum_{p,q=1}^{2n} b_{pq} x_p x_q} \right\} \right\} = \operatorname*{extrem}_{\substack{a_{pq} \in a^{\mathrm{I}}_{pq}, b_{pq} \in b^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \left\{ \mathrm{Re}\left\{ \frac{R_1}{R_2} \right\} \right\} \tag{27}$$

According to the optimum theory, the extreme value problem $R = \operatorname*{extrem}_{\substack{a_{pq} \in a^{\mathrm{I}}_{pq}, b_{pq} \in b^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \{\mathrm{Re}\{R_1/R_2\}\}$ can be decomposited into the real part of the quotient of the two extreme value problems $\operatorname*{extrem}_{\substack{a_{pq} \in a^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \left\{ \mathrm{Re}\{R_1\} + \sqrt{-1} \left(\operatorname*{extrem}_{\substack{a_{pq} \in a^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \{\mathrm{Im}\{R_1\}\} \right) \right\}$ and $\operatorname*{extrem}_{\substack{b_{pq} \in b^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \left\{ \mathrm{Re}\{R_2\} + \sqrt{-1} \left(\operatorname*{extrem}_{\substack{b_{pq} \in b^{\mathrm{I}}_{pq} \\ p,q=1,2,\ldots,2n}} \{\mathrm{Im}\{R_2\}\} \right) \right\}$, i.e.

$$R = \underset{\substack{a_{pq}\in a^{\mathrm{I}}_{pq}, b_{pq}\in b^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \left\{ \mathrm{Re}\left\{\frac{R_1}{R_2}\right\}\right\} = \mathrm{Re}\left\{ \frac{\underset{\substack{a_{pq}\in a^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Re}\{R_1\} + \sqrt{-1}\left(\underset{\substack{a_{pq}\in a^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Im}\{R_1\}\right)}{\underset{\substack{b_{pq}\in b^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Re}\{R_2\} + \sqrt{-1}\left(\underset{\substack{b_{pq}\in b^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Im}\{R_2\}\right)} \right\} \tag{28}$$

We can know that the quantity R_1 is linear function of the elements a_{pq}, $p,q = 1,2,\ldots,2n$. Based on the extreme theorem in convex analysis, since the real part and imaginary part of the quantity R_1 are all convex (or concave) functions of the elements a_{pq}, $p,q = 1,2,\ldots,2n$, and the interval sets $a^{\mathrm{I}}_{pq} = [\underline{a}_{pq}, \overline{a}_{pq}]$, $p,q = 1,2,\ldots,2n$ are all convex, the extreme values of the real part and imaginary part of R_1 will be reached on the boundary matrices or vertex matrices of the interval matrices $A^{\mathrm{I}} = [\underline{A}, \overline{A}] = (a^{\mathrm{I}}_{ij})$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}] = (b^{\mathrm{I}}_{ij})$, i.e.

$$\begin{aligned}&\underset{\substack{a_{pq}\in a^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Re}\left\{\sum_{p,q=1}^{2n} a_{pq}x_p x_q\right\} + \sqrt{-1}\left(\underset{\substack{a_{pq}\in a^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Im}\left\{\sum_{p,q=1}^{2n} a_{pq}x_p x_q\right\}\right)\\ &\quad = \mathrm{Re}\left\{\sum_{p,q=1}^{2n} \hat{a}^{s}_{pq}x_p x_q\right\} + \sqrt{-1}\left(\mathrm{Im}\left\{\sum_{p,q=1}^{2n} \hat{a}^{s}_{pq}x_p x_q\right\}\right) = \mathrm{Re}\{x^{\mathrm{T}}\hat{A}_s x\} + \sqrt{-1}\left(\mathrm{Im}\{x^{\mathrm{T}}\hat{A}_s x\}\right)\end{aligned} \tag{29}$$

Similarly, we also have that

$$\begin{aligned}&\underset{\substack{b_{pq}\in b^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Re}\left\{\sum_{p,q=1}^{2n} b_{pq}x_p x_q\right\} + \sqrt{-1}\left(\underset{\substack{b_{pq}\in b^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \mathrm{Im}\left\{\sum_{p,q=1}^{2n} b_{pq}x_p x_q\right\}\right)\\ &\quad = \mathrm{Re}\left\{\sum_{p,q=1}^{2n} \hat{b}^{t}_{pq}x_p x_q\right\} + \sqrt{-1}\left(\mathrm{Im}\left\{\sum_{p,q=1}^{2n} \hat{b}^{t}_{pq}x_p x_q\right\}\right) = \mathrm{Re}\{x^{\mathrm{T}}\hat{B}_t x\} + \sqrt{-1}\left(\mathrm{Im}\{x^{\mathrm{T}}\hat{B}_t x\}\right)\end{aligned} \tag{30}$$

Thus, we can obtain

$$R = \underset{\substack{a_{pq}\in a^{\mathrm{I}}_{pq}, b_{pq}\in b^{\mathrm{I}}_{pq}\\ p,q=1,2,\ldots,2n}}{\text{extrem}} \left\{ \mathrm{Re}\left\{\frac{\sum_{p,q=1}^{2n} a_{pq}x_p x_q}{\sum_{p,q=1}^{2n} b_{pq}x_p x_q}\right\}\right\} = \mathrm{Re}\left\{\frac{x^{\mathrm{T}}\hat{A}_s x}{x^{\mathrm{T}}\hat{B}_t x}\right\} \tag{31}$$

Substitution of Eq. (31) into Eq. (26) yielding

$$\lambda_{rist} = \lambda_{ri\mathrm{ext}}(\langle\hat{A}_s, \hat{B}_t\rangle) = \min_{S_i\subset C^n} \max_{\substack{x\neq 0\\ x\in S_i}} \left\{\mathrm{Re}\left\{\frac{x^{\mathrm{T}}\hat{A}_s x}{x^{\mathrm{T}}\hat{B}_t x}\right\}\right\} \quad s,t = 1,2,3,\ldots,2^{2n\times 2n},\ i = 1,2,\ldots,2n \tag{32}$$

Thus, the maximum and minimum values of the real parts λ_{ri}, $i = 1,2,\ldots,2n$ of the complex eigenvalues λ_i, $i = 1,2,\ldots,2n$ can, respectively, be determined by

$$\overline{\lambda}_{ri} = \lambda_{ri\max} = \max_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{rist}\} = \max_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \left\{\lambda_{ri\mathrm{ext}}(\langle\hat{A}_s, \hat{B}_t\rangle)\right\}, \quad i = 1,2,\ldots,2n \tag{33}$$

and

$$\underline{\lambda}_{ri} = \lambda_{ri\max} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{rist}\} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \left\{\lambda_{ri\mathrm{ext}}(\langle\hat{A}_s, \hat{B}_t\rangle)\right\}, \quad i = 1,2,\ldots,2n \tag{34}$$

In the same manner, the maximum and minimum values of the imaginary parts λ_{yi}, $i = 1,2,\ldots,2n$ of the complex eigenvalues λ_i, $i = 1,2,\ldots,2n$, can, respectively, be determined by

$$\overline{\lambda}_{yi} = \lambda_{yi\max} = \max_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{yist}\} = \max_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{yiext}(\langle \hat{A}_s, \hat{B}_t\rangle)\}, \quad i = 1,2,\ldots,2n \tag{35}$$

and

$$\underline{\lambda}_{yi} = \lambda_{yi\max} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{yist}\} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{yiext}(\langle \hat{A}_s, \hat{B}_t\rangle)\}, \quad i = 1,2,\ldots,2n \tag{36}$$

The stationary value problem of Rayleigh's quotient of the matrix pair $\langle \hat{A}_s, \hat{B}_t\rangle$ is equivalent to its algebraic eigenvalue problem. Thus, the eigenvalue problem corresponding to Eq. (10) reads

$$\hat{A}_s x_{ist} = \lambda_{ist}\hat{B}_t x_{ist}, \quad s,t = 1,2,3,\ldots,2^{2n\times 2n}, \ \ i = 1,2,\ldots,2n \tag{37}$$

where $\hat{A}_s = (\hat{a}^s_{ij})$ and $\hat{B}_t = (\hat{b}^t_{ij})$, the vector x_{ist} is the eigenvector associated with the ith eigenvalue λ_{ist}.

Thus, we arrive at the following:

3.1. Vertex solution theorem

If the interval matrix $A^{\mathrm{I}} = [\underline{A}, \overline{A}] = (a^{\mathrm{I}}_{ij})$ is real symmetric and non-positive definite, and its boundary matrix or vertex matrix is expressed as $\hat{A}_s = (\hat{a}^s_{ij})$, where $\hat{a}^s_{ij} = \hat{a}^s_{ji} = \underline{a}_{ij}$ (or $\overline{a}_{ij}$), $i,j = 1,2,\ldots,2n$, $s = 1,2,3,\ldots,2^{2n\times 2n}$, and the interval matrix $B^{\mathrm{I}} = [\underline{B}, \overline{B}] = (b^{\mathrm{I}}_{ij})$ is also real symmetric and non-positive definite, and its boundary matrix or vertex matrix is expressed as $\hat{B}_t = (\hat{b}^t_{ij})$, where $\hat{b}^t_{ij} = \hat{b}^t_{ji} = \underline{b}_{ij}$ (or $\overline{b}_{ij}$), $i,j = 1,2,\ldots,2n$, $t = 1,2,3,\ldots,2^{2n\times 2n}$. Then the real part $\lambda^{\mathrm{I}}_{ri}, i = 1,2,\ldots,2n$ of the interval complex eigenvalues $\lambda^{\mathrm{I}}_i = \lambda^{\mathrm{I}}_{ri} + \sqrt{-1}\lambda^{\mathrm{I}}_{yi}$, $i = 1,2,\ldots,2n$, of the interval matrix pair $\langle A^{\mathrm{I}}, B^{\mathrm{I}}\rangle$ can be determined as follows:

$$\lambda^{\mathrm{I}}_{ri} = [\underline{\lambda}_{ri}, \overline{\lambda}_{ri}], \quad i = 1,2,\ldots,2n \tag{38}$$

where the upper bound $\overline{\lambda}_{ri}, i = 1,2,\ldots,2n$, and the lower bound $\underline{\lambda}_{ri}$, $i = 1,2,\ldots,2n$ of the real part of the interval complex eigenvalues $\lambda^{\mathrm{I}}_i = \lambda^{\mathrm{I}}_{ri} + \sqrt{-1}\lambda^{\mathrm{I}}_{yi}, i = 1,2,\ldots,2n$, can be obtained by

$$\overline{\lambda}_{ri} = \alpha_{ri\max} = \max_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{rist}(\langle \hat{A}_s, \hat{B}_t\rangle)\} = \max_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\mathrm{Re}(\lambda_{ist}(\langle \hat{A}_s, \hat{B}_t\rangle))\} \quad i = 1,2,\ldots,2n \tag{39a}$$

and

$$\underline{\lambda}_{ri} = \lambda_{ri\min} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{rist}(\langle \hat{A}_s, \hat{B}_t\rangle)\} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\mathrm{Re}(\lambda_{ist}(\langle \hat{A}_s, \hat{B}_t\rangle))\} \quad i = 1,2,\ldots,2n \tag{39b}$$

and the imaginary part $\lambda^{\mathrm{I}}_{yi}, i = 1,2,\ldots,2n$ of the interval complex eigenvalues $\lambda^{\mathrm{I}}_i = \lambda^{\mathrm{I}}_{ri} + \sqrt{-1}\lambda^{\mathrm{I}}_{yi}$, $i = 1,2,\ldots,2n$, of the real non-positive definite interval matrix pair $\langle A^{\mathrm{I}}, B^{\mathrm{I}}\rangle$can be determined as follows

$$\lambda^{\mathrm{I}}_{yi} = [\underline{\lambda}_{yi}, \overline{\lambda}_{yi}], \quad i = 1,2,\ldots,2n \tag{40}$$

where the upper bound $\overline{\lambda}_{yi}, i = 1,2,\ldots,2n$, and the lower bound $\underline{\lambda}_{yi}$, $i = 1,2,\ldots,2n$ of the imaginary part of the interval complex eigenvalues $\lambda^{\mathrm{I}}_i = \lambda^{\mathrm{I}}_{ri} + \sqrt{-1}\lambda^{\mathrm{I}}_{yi}, i = 1,2,\ldots,2n$, can be obtained by

$$\overline{\lambda}_{yi} = \lambda_{yi\max} = \max_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{yist}(\langle \hat{A}_s, \hat{B}_t\rangle)\} = \max\nolimits_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\mathrm{Im}(\lambda_{ist}(\langle \hat{A}_s, \hat{B}_t\rangle))\} \quad i = 1,2,\ldots,2n \tag{41a}$$

and

$$\underline{\lambda}_{yi} = \lambda_{yi\min} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\lambda_{yist}(\langle \hat{A}_s, \hat{B}_t\rangle)\} = \min_{1\leqslant s,t\leqslant 2^{2n\times 2n}} \{\mathrm{Im}(\lambda_{ist}(\langle \hat{A}_s, \hat{B}_t\rangle))\} \quad i = 1,2,\ldots,2n \tag{41b}$$

where $\lambda_{ist} = \lambda_{rist} + \sqrt{-1}\lambda_{yist}, i = 1, 2, \ldots, 2n; s, t = 1, 2, 3, \ldots, 2^{2n\times 2n}$, satisfy the following generalized eigenvalue problems:

$$\hat{A}_s u_{ist} = \lambda_{ist}\hat{B}_t u_{ist}, \quad s, t = 1, 2, \ldots, 2^{2n\times 2n}, \ \ i = 1, 2, \ldots, 2n \tag{42}$$

where u_{ist} is the eigenvector associated with the ith eigenvalue λ_{ist}.

4. Interval perturbation method

By the virtue of the central notation of the interval matrix for the interval matrices $A^{\mathrm{I}} = [\underline{A}, \overline{A}]$ and $B^{\mathrm{I}} = [\underline{B}, \overline{B}]$, we have

$$A^{\mathrm{I}} = A^{\mathrm{c}} + \Delta A^{\mathrm{I}}, \quad B^{\mathrm{I}} = B^{\mathrm{c}} + \Delta B^{\mathrm{I}} \tag{43}$$

where $\Delta A^{\mathrm{I}} = [-\Delta A, \Delta A]$ and $\Delta B^{\mathrm{I}} = [-\Delta B, \Delta B]$.

Thus, by Eq. (43), the generalized interval eigenvalue problem Eq. (10) can be written in the following useful form:

$$(A^{\mathrm{c}} + \Delta A^{\mathrm{I}})x = \lambda(B^{\mathrm{c}} + \Delta B^{\mathrm{I}})x \tag{44a}$$

and

$$y^{\mathrm{T}}(A^{\mathrm{c}} + \Delta A^{\mathrm{I}}) = \lambda y^{\mathrm{T}}(B^{\mathrm{c}} + \Delta B^{\mathrm{I}}) \tag{44b}$$

In this study, based on Eq. (44), we will present an interval perturbation method for the generalized interval eigenvalue problem.

For the sake of completeness, we review the matrix perturbation theory. The purpose of the perturbation theory is to show approximately how the eigenvalues change as the matrix changes.

Let us consider any $n \times n$ real symmetric matrix pair A_0 and B_0, and let us denote its eigenvalues by λ_{0i}, $i = 1,2,\ldots,n$, its right eigenvector by x_{0i}, $i = 1,2,\ldots,n$ and its eigenvectors by y_{0i}, $i = 1,2,\ldots,n$, where the eigenvalues and eigenvectors satisfy

$$A_0^{\mathrm{c}} x_{0i} = \lambda_{0i} B_0^{\mathrm{c}} x_{0i}, \quad y_{0i}^{\mathrm{T}} A_0^{\mathrm{c}} = \lambda_{0i} y_{0i}^{\mathrm{T}} B_0^{\mathrm{c}}, \ \ i = 1, 2, \ldots, n \tag{45}$$

For convenience, we shall assume that the eigenvectors x_{0i}, $i = 1,2,\ldots,n$ and y_{0i}, $i = 1,2,\ldots,n$ have been normalized so as to satisfy

$$y_{0j}^{\mathrm{T}} A_0^{\mathrm{c}} x_{0i} = \lambda_{0i}\delta_{ij}, \quad y_{0i}^{\mathrm{T}} B_0 x_{0i} = \delta_{ij}, \ \ i = 1, 2, \ldots, n \tag{46}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad i, j = 1, 2, \ldots, n \tag{47}$$

If small changes are introduced to the matrices A and B, i.e.

$$A = A_0 + \delta A, \quad B = B_0 + \delta B \tag{48}$$

where A_0 and B_0 are the unperturbed matrix pair, and δA and δB are the matrix pair representing the small changes from A_0 and B_0. We shall refer to A and B as the perturbed matrix pair. The perturbed generalized eigenvalue problem can be written in the form

$$(A_0 + \delta A)x_i = \lambda_i(B_0 + \delta B)x_i, \quad y_i^{\mathrm{T}}(A_0 + \delta A) = \lambda_i y_i^{\mathrm{T}}(B_0 + \delta B), \ \ i = 1, 2, \ldots, n \tag{49}$$

where λ_i, $i = 1,2,\ldots,n$ are the perturbed eigenvalues, x_i, $i = 1,2,\ldots,n$ the perturbed right eigenvectors and y_i, $i = 1,2,\ldots,n$ the perturbed left eigenvectors. As to the unperturbed problem, the eigenvalues are assumed to be distinct and the eigenvectors bi-orthonormal, i.e.,

$$y_j^{\mathrm{T}} A x_i = \lambda_i \delta_{ij}, \quad y_i^{\mathrm{T}} B x_i = \delta_{ij}, \quad i = 1,2,\ldots,n \tag{50}$$

Assuming that A_0, B_0, δA, δB, λ_{0i}, $i = 1,2,\ldots,n$, x_{0i}, $i = 1,2,\ldots,n$, and y_{0i}, $i = 1,2,\ldots,n$ are known, and λ_{0i}, $i = 1,2,\ldots,n$ are all distinct eigenvalues, one can obtain the expression of the first perturbation eigenvalues for the perturbed matrix pair $A = A_0 + \delta A$ and $B = B_0 + \delta B$ as follows:

$$\lambda_i = \lambda_{0i} + \delta\lambda_i = \lambda_{0i} + y_{0i}^{\mathrm{T}}(\delta A - \lambda_{0i}\delta B)x_{0i}, \quad i = 1,2,\ldots,n \tag{51}$$

where

$$\delta\lambda_i = y_{0i}^{\mathrm{T}}(\delta A - \lambda_{0i}\delta B)x_{0i} \tag{52}$$

Since $\lambda_i = \lambda_{ir} + \sqrt{-1}\lambda_{iy}$, $\lambda_{0i} = \lambda_{0ir} + \sqrt{-1}\lambda_{0iy}$, $x_{0i} = x_{0ir} + \sqrt{-1}x_{0iy}$ and $y_{0i} = y_{0ir} + \sqrt{-1}y_{0iy}$, $i = 1,2,\ldots,n$ are the complex numbers or complex vectors, and Eq. (51) can be written in real and imaginary part forms

$$\lambda_i = \lambda_{ir} + \sqrt{-1}\lambda_{iy} = (\lambda_{0ir} + \delta\lambda_{ir}) + \sqrt{-1}(\lambda_{0iy} + \delta\lambda_{iy}), \quad i = 1,2,\ldots,n \tag{53}$$

where

$$\lambda_{ir} = \lambda_{0ir} + \delta\lambda_{ir}, \quad \lambda_{iy} = \lambda_{0iy} + \delta\lambda_{iy}, \quad i = 1,2,\ldots,n \tag{54}$$

where

$$\delta\lambda_{ir} = y_{0ir}^{\mathrm{T}}\delta A x_{0ir} - y_{0iy}^{\mathrm{T}}\delta A x_{0iy} + \lambda_{0ir}(y_{0ir}^{\mathrm{T}}\delta B x_{0ir} - y_{0iy}^{\mathrm{T}}\delta B x_{0iy}) - \lambda_{0iy}(y_{0ic}^{\mathrm{T}}\delta B x_{0iy} + y_{0iy}^{\mathrm{T}}\delta B x_{0ir}) \tag{55}$$

and

$$\delta\lambda_{iy} = y_{0ir}^{\mathrm{T}}\delta A x_{0iy} + y_{0iy}^{\mathrm{T}}\delta A x_{0ir} + \lambda_{0iy}(y_{0ir}^{\mathrm{T}}\delta B x_{0ir} - y_{0iy}^{\mathrm{T}}\delta B x_{0iy}) + \lambda_{0ir}(y_{0ir}^{\mathrm{T}}\delta B x_{0iy} + y_{0iy}^{\mathrm{T}}\delta B x_{0ir}) \tag{56}$$

Consider the generalized interval eigenvalue problem Eq. (10). Under the small deviation amplitudes $\Delta A^{\mathrm{I}} = [-\Delta A, \Delta A]$ and $\Delta B^{\mathrm{I}} = [-\Delta B, \Delta B]$ of the interval matrix pair $A^{\mathrm{I}} = A^{\mathrm{c}} + \Delta A^{\mathrm{I}}$ and $B^{\mathrm{I}} = B^{\mathrm{c}} + \Delta B^{\mathrm{I}}$, if we view $\Delta A^{\mathrm{I}} = [-\Delta A, \Delta A]$ and $\Delta B^{\mathrm{I}} = [-\Delta B, \Delta B]$ as interval perturbations around the matrix pair A^{c} and B^{c}, we can solve the generalized interval eigenvalue problem by the matrix perturbation method. By the interval mathematics or interval analysis, from the first equation of Eq. (54), we can obtain the interval extension of the real part of the complex eigenvalues λ_i, $i = 1,2,\ldots,n$

$$\lambda_{ir}^{\mathrm{I}} = [\underline{\lambda}_{ir}, \overline{\lambda}_{ir}] = \lambda_{cir} + \delta\lambda_{ir}^{\mathrm{I}}, \quad i = 1,2,\ldots,n \tag{57}$$

where

$$\begin{aligned}\delta\lambda_{ir}^{\mathrm{I}} &= [\delta\underline{\lambda}_{ir}, \delta\overline{\lambda}_{ir}] \\ &= y_{cir}^{\mathrm{T}}\Delta A^{\mathrm{I}} x_{cir} - y_{ciy}^{\mathrm{T}}\Delta A^{\mathrm{I}} x_{ciy} + \lambda_{cir}(y_{cir}^{\mathrm{T}}\Delta B^{\mathrm{I}} x_{cir} - y_{ciy}^{\mathrm{T}}\Delta B^{\mathrm{I}} x_{ciy}) - \lambda_{ciy}(y_{cic}^{\mathrm{T}}\Delta B^{\mathrm{I}} x_{ciy} + y_{ciy}^{\mathrm{T}}\Delta B^{\mathrm{I}} x_{cir})\end{aligned} \tag{58}$$

Substituting Eq. (58) into Eq. (57), by the interval operations, gives

$$\lambda_{ir}^{\mathrm{I}} = [\underline{\lambda}_{ir}, \overline{\lambda}_{ir}] = [\lambda_{cir} + \delta\underline{\lambda}_{ir}, \lambda_{cir} + \delta\overline{\lambda}_{ir}], \quad i = 1,2,\ldots,n \tag{59}$$

where

$$\overline{\lambda}_{ir} = \lambda_{cir} + \delta\overline{\lambda}_{ir} = \lambda_{cir} + \left| \begin{array}{c} y_{cir}^{\mathrm{T}}\Delta A x_{cir} - y_{cir}^{\mathrm{T}}\Delta A x_{cir} + \\ \lambda_{cir}(y_{cir}^{\mathrm{T}}\Delta B x_{cir} - y_{ciy}^{\mathrm{T}}\Delta B x_{ciy}) - \\ \lambda_{ciy}(y_{cir}^{\mathrm{T}}\Delta B x_{ciy} + y_{ciy}^{\mathrm{T}}\Delta B x_{cir}) \end{array} \right|, \quad i = 1,2,\ldots,n \tag{60a}$$

and

$$\underline{\lambda}_{ir} = \lambda_{cir} + \delta\underline{\lambda}_{ir} = \lambda_{cir} - \left| \begin{array}{c} y_{cir}^{\mathrm{T}}\Delta A x_{cir} - y_{cir}^{\mathrm{T}}\Delta A x_{cir} + \\ \lambda_{cir}(y_{cir}^{\mathrm{T}}\Delta B x_{cir} - y_{ciy}^{\mathrm{T}}\Delta B x_{ciy}) - \\ \lambda_{ciy}(y_{cir}^{\mathrm{T}}\Delta B x_{ciy} + y_{ciy}^{\mathrm{T}}\Delta B x_{cir}) \end{array} \right|, \quad i = 1, 2, \ldots, n \tag{60b}$$

Let

$$\Delta\lambda_{iy} = \left| \begin{array}{c} y_{cir}^{\mathrm{T}}\Delta A x_{cir} - y_{cir}^{\mathrm{T}}\Delta A x_{cir} + \\ \lambda_{cir}(y_{cir}^{\mathrm{T}}\Delta B x_{cir} - y_{ciy}^{\mathrm{T}}\Delta B x_{ciy}) - \\ \lambda_{ciy}(y_{cir}^{\mathrm{T}}\Delta B x_{ciy} + y_{ciy}^{\mathrm{T}}\Delta B x_{cir}) \end{array} \right| \tag{61}$$

then Eq. (60) becomes

$$\overline{\lambda}_{ir} = \lambda_{cir} + \Delta\lambda_{ir}, \underline{\lambda}_{ir} = \lambda_{cir} - \Delta\lambda_{ir}, \quad i = 1, 2, \ldots, n \tag{62}$$

In the similar manner, we also can obtain the imaginary part of the complex eigenvalues λ_i, $i = 1, 2, \ldots, n$

$$\lambda_{ir}^{\mathrm{I}} = [\underline{\lambda}_{ir}, \overline{\lambda}_{ir}], \quad i = 1, 2, \ldots, n \tag{63}$$

where

$$\overline{\lambda}_{iy} = \lambda_{ciy} + \Delta\lambda_{iy}, \quad \underline{\lambda}_{iy} = \lambda_{ciy} - \Delta\lambda_{iy}, \ \ i = 1, 2, \ldots, n \tag{64}$$

in which

$$\Delta\lambda_{iy} = \left| \begin{array}{c} y_{cir}^{\mathrm{T}}\Delta A x_{ciy} - y_{ciy}^{\mathrm{T}}\Delta A x_{cir} + \\ \lambda_{ciy}(y_{cir}^{\mathrm{T}}\Delta B x_{cir} - y_{ciy}^{\mathrm{T}}\Delta B x_{ciy}) - \\ \lambda_{cir}(y_{cir}^{\mathrm{T}}\Delta B x_{ciy} + y_{ciy}^{\mathrm{T}}\Delta B x_{cir}) \end{array} \right| \tag{65}$$

In Eqs. (62) and (64), the eigenvalues $\lambda_{ci} = \lambda_{cir} + \sqrt{-1}\lambda_{ciy}, i = 1, 2, \ldots, n$ and the eigenvectors $x_{ci} = x_{cir} + \sqrt{-1}x_{ciy}, i = 1, 2, \ldots, n$ and $y_{ci} = y_{cir} + \sqrt{-1}y_{ciy}, i = 1, 2, \ldots, n$ satisfy

$$y_{cj}^{\mathrm{T}} A^{\mathrm{c}} x_{ci} = \lambda_{ci}\delta_{ij}, \quad y_{cj}^{\mathrm{T}} B^{\mathrm{c}} x_{ci} = \delta_{ij}, \ \ i = 1, 2, \ldots, n \tag{66}$$

Thus, we arrive at the following solution theorem for the generalized interval eigenvalue problem.

4.1. Perturbation solution theorem

If $\lambda_{ci} = \lambda_{cir} + \sqrt{-1}\lambda_{ciy}, \ i = 1, 2, \ldots, n$, are all distinct eigenvalues of the midpoint matrix pair A^{c} and B^{c} of real symmetric non-positive definite interval matrix pair $A^{\mathrm{I}} = A^{\mathrm{c}} + \Delta \ A^{\mathrm{I}}$ and $B^{\mathrm{I}} = B^{\mathrm{c}} + \Delta B^{\mathrm{I}}$, which $\Delta A^{\mathrm{I}} = [-\Delta A, \Delta \ A]$ and $\Delta B^{\mathrm{I}} = [-\Delta B, \Delta B]$ are small uncertainty matrices, with the corresponding right eigenvectors $x_{ci} = x_{cir} + \sqrt{-1}x_{ciy}, \ i = 1, 2, \ldots, n$,and the left eigenvectors $y_{ci} = y_{cir} + \sqrt{-1}y_{ciy}, \ i = 1, 2, \ldots, n$. Then the interval eigenvalues of interval matrix pair $A^{\mathrm{I}} = A^{\mathrm{c}} + \Delta A^{\mathrm{I}}$ and $B^{\mathrm{I}} = B^{\mathrm{c}} + \Delta B^{\mathrm{I}}$ are given by the first-order perturbation as follows:

$$\lambda_i^{\mathrm{I}} = \lambda_{ir}^{\mathrm{I}} + \sqrt{-1}\lambda_{iy}^{\mathrm{I}}, \quad \lambda_{ir}^{\mathrm{I}} = [\underline{\lambda}_{ir}, \overline{\lambda}_{ir}], \quad \lambda_{iy}^{\mathrm{I}} = [\underline{\lambda}_{iy}, \overline{\lambda}_{iy}], \ \ 1, 2, \ldots, n \tag{67}$$

where

$$\overline{\lambda}_{ir} = \lambda_{cir} + \Delta\lambda_{ir}, \quad \underline{\lambda}_{ir} = \lambda_{cir} - \Delta\lambda_{ir}, \ \ i = 1, 2, \ldots, n \tag{68a}$$

and

$$\overline{\lambda}_{iy} = \lambda_{ciy} + \Delta\lambda_{iy}, \quad \underline{\lambda}_{iy} = \lambda_{ciy} - \Delta\lambda_{iy}, \ \ i = 1, 2, \ldots, n \tag{68b}$$

where

$$\Delta\lambda_{ir} = \left| \begin{array}{c} y_{cir}^{\mathrm{T}}\Delta A x_{cir} - y_{cir}^{\mathrm{T}}\Delta A x_{cir} + \\ \lambda_{cir}(y_{cir}^{\mathrm{T}}\Delta B x_{cir} - y_{ciy}^{\mathrm{T}}\Delta B x_{ciy}) - \\ \lambda_{ciy}(y_{cir}^{\mathrm{T}}\Delta B x_{ciy} + y_{ciy}^{\mathrm{T}}\Delta B x_{cir}) \end{array} \right|, \quad \Delta\lambda_{iy} = \left| \begin{array}{c} y_{cir}^{\mathrm{T}}\Delta A x_{ciy} - y_{ciy}^{\mathrm{T}}\Delta A x_{cir} + \\ \lambda_{ciy}(y_{cir}^{\mathrm{T}}\Delta B x_{cir} - y_{ciy}^{\mathrm{T}}\Delta B x_{ciy}) - \\ \lambda_{cir}(y_{cir}^{\mathrm{T}}\Delta B x_{ciy} + y_{ciy}^{\mathrm{T}}\Delta B x_{cir}) \end{array} \right| \tag{69}$$

in which the eigenvalues $\lambda_{ci} = \lambda_{cir} + \sqrt{-1}\lambda_{ciy}, i = 1, 2, \ldots, n$, the right eigenvectors $x_{ci} = x_{cir} + \sqrt{-1}x_{ciy}$, $i = 1, 2, \ldots, n$ and the left eigenvectors $y_{ci} = y_{cir} + \sqrt{-1}y_{ciy},\ i = 1, 2, \ldots, n$ satisfy

$$y_{cj}^{\mathrm{T}}A^{\mathrm{c}}x_{ci} = \lambda_{ci}\delta_{ij}, \quad y_{cj}^{\mathrm{T}}B^{\mathrm{c}}x_{ci} = \delta_{ij}, \ \ i = 1, 2, \ldots, n \tag{70}$$

Obviously, from the Eq. (68), we can see that we only need to solve two generalized eigenvalue problems and compute four expressions, then all interval eigenvalues of the interval matrix can be determined. Thus, the presented solution theorem is very practical.

5. Numerical example

In this section, the proposed vertex solution theorem and interval perturbation method are applied to a seven degree-of-freedom spring-damping-mass system with uncertainty to illustrate its effectiveness as shown in Fig. 1. It is assumed that due to the manufacture errors or the unavoidable scatters in the material properties, a part of physical quantities of the system exhibit some uncertainties. For convenience, all the quantities are dimensionless. In this numerical example it is assumed that m_1, m_2, C_1, C_2, K_5 and K_6 are uncertain-but-bounded variables, and the interval masses are taken as $m_i = [m_i^{\mathrm{c}} - \beta m_i^{\mathrm{c}}, m_i^{\mathrm{c}} + \beta m_i^{\mathrm{c}}]$, $i = 1, 2$, interval damping $C_i = [C_i^{\mathrm{c}} - \beta C_i^{\mathrm{c}}, C_i^{\mathrm{c}} + \beta C_i^{\mathrm{c}}], i = 1, 2$, interval spring constants $K_i = [K_i^{\mathrm{c}} - \beta K_i^{\mathrm{c}}, K_i^{\mathrm{c}} + \beta K_i^{\mathrm{c}}], i = 5, 6$, where $m_i^{\mathrm{c}} = 2.0(i = 1, 2), C_i^{\mathrm{c}} = 0.03(i = 1, 2) K_i^{\mathrm{c}} = 1.0(i = 5, 6)$ and β is the variable parameter. Other quantities are deterministic, in which masses are taken as $m_i = 2.0$, $i = 3, 4, 5, 6$, $m_7 = 1.0$, damping $C_i = 0.02$, $i = 3, 4 C_i = 0.01$, $i = 5, 6, 7, 8$, spring constants $K_i = 1.0$, $i = 3, 4, \ldots, 8$. In the following the regions of the real part and imaginary part of complex eigenvalues are calculated using the presented vertex solution method and the interval perturbation method in comparison with Deif's solution theorem (Deif, 1991).

The real part and imaginary part of complex eigenvalues computed by the vertex solution theorem and Deif's solution method are listed in Tables 1 and 2 respectively when the variable parameter β is taken as 0.01. In the tables k is the number of modes; λ_r^{c} and λ_y^{c} are the real part and the imaginary part of the nominal complex eigenvalue, respectively; $\overline{\lambda}_r^{\mathrm{V}}$ and $\underline{\lambda}_r^{\mathrm{V}}$ are the upper bound and lower bound of the real part of the complex eigenvalues using the vertex solution theorem, respectively; $\overline{\lambda}_y^{\mathrm{V}}$ and $\underline{\lambda}_y^{\mathrm{V}}$ are the upper bound and lower bound of the imaginary part of the complex eigenvalues using the vertex solution theorem, respectively; $\overline{\lambda}_r^{\mathrm{D}}$ and $\underline{\lambda}_r^{\mathrm{D}}$ are the upper bound and lower bound of the real part of the complex eigenvalues obtained by Deif's method, respectively; $\overline{\lambda}_y^{\mathrm{D}}$ and $\underline{\lambda}_y^{\mathrm{D}}$ are the upper bound and lower bound of the imaginary part of the complex eigenvalues obtained by Deif's method, respectively.

From the results listed in the Tables 1 and 2, we can see that the maximum or upper bounds and the minimum or lower bounds on the real part and the imaginary part of complex eigenvalues yielded by the vertex solution theorem are the same as those produced by Deif's solution theorem. Nevertheless,

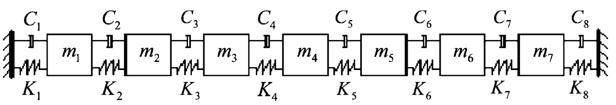

Fig. 1. A spring-damping-mass system with seven degrees of freedom.

Table 1
Lower and upper bounds of the real part of complex eigenvalues ($\beta = 0.01$)

k	λ_r^c	$\underline{\lambda}_r^V$	$\bar{\lambda}_r^V$	$\bar{\lambda}_r^D$	$\underline{\lambda}_r^D$
1(2)	−0.747623E−03	−0.798942E−03	−0.703841E−03	−0.798942E−03	−0.703841E−03
3(4)	−0.280321E−02	−0.290725E−02	−0.270265E−02	−0.290725E−02	−0.270265E−02
5(6)	−0.582838E−02	−0.596573E−02	−0.569539E−02	−0.596573E−02	−0.569539E−02
7(8)	−0.108606E−01	−0.110266E−01	−0.106931E−01	−0.110266E−01	−0.106931E−01
9(10)	−0.144704E−01	−0.147770E−01	−0.141723E−01	−0.147770E−01	−0.141723E−01
11(12)	−0.181753E−01	−0.186466E−01	−0.177204E−01	−0.186466E−01	−0.177204E−01
13(14)	−0.121144E−01	−0.121179E−01	−0.121110E−01	−0.121179E−01	−0.121110E−01

Table 2
Lower and upper bounds of the imaginary part of complex eigenvalues ($\beta = 0.01$)

k	λ_y^c	$\underline{\lambda}_y^V$	$\bar{\lambda}_y^V$	$\bar{\lambda}_y^D$	$\underline{\lambda}_y^D$
1	0.278316	0.262973	0.292354	0.262973	0.292354
2	−0.278316	−0.292354	−0.262973	−0.292354	−0.262973
3	0.556033	0.550272	0.561972	0.550272	0.561972
4	−0.556033	−0.561972	−0.550272	−0.561972	−0.550272
5	0.820673	0.817410	0.823958	0.817410	0.823958
6	−0.820673	−0.823958	−0.817410	−0.823958	−0.817410
7	1.054730	1.050560	1.058913	1.050560	1.058913
8	−1.054730	−1.058913	−1.050560	−1.058913	−1.050560
9	1.241950	1.236371	1.247545	1.236371	1.247545
10	−1.241950	−1.247545	−1.236371	−1.247545	−1.236371
11	1.367428	1.363960	1.370954	1.363960	1.370954
12	−1.367428	−1.370954	−1.363960	−1.370954	−1.363960
13	1.553723	1.552818	1.554650	1.552818	1.554650
14	−1.553723	−1.554650	−1.552818	−1.554650	−1.552818

the presented method is superior to Deif's method because there exists many difficulties in Deif's solution theorem, such as: it is quite difficult how to determine the invariance properties of the eigenvectors' components in the interval matrix; large computational efforts etc.

Comparisons of the range curves of the real part of the complex eigenvalues of the system computed by the vertex solution theorem and the interval perturbation method when the variable parameter β ranges from 0.00 to 0.05 are plotted in Fig. 2(a)–(g), and the imaginary part plotted in Fig. 3(a)–(n). $\bar{\lambda}_r^P$ and $\underline{\lambda}_r^P$ denote the upper bound and lower bound of the real part of the complex eigenvalues obtained by the interval perturbation method, respectively; $\bar{\lambda}_y^P$ and $\underline{\lambda}_y^P$ denote the upper bound and lower bound of the imaginary part of the complex eigenvalues obtained by the interval perturbation method respectively.

It can be seen from the Figs. 2 and 3 that as far as the real part and the imaginary part of complex eigenvalues is concerned, the low order eigenvalues obtained by the interval perturbation are contained by those yielded by the vertex solution method. That is to say, the lower bounds within the vertex solution method are smaller than those predicted by the interval perturbation method. Likewise, the upper bounds furnished by the vertex solution method are larger than those yielded by the interval perturbation method. However, for the high order eigenvalues, the region curves produced by the interval perturbation method gradually approach those obtained by the vertex solution method. As far the imaginary part of complex eigenvalues, sometimes they even arrive to coincidence; sometimes instead the width of the region bounds obtained by the interval perturbation method is slightly larger than that yielded by the vertex solution method.

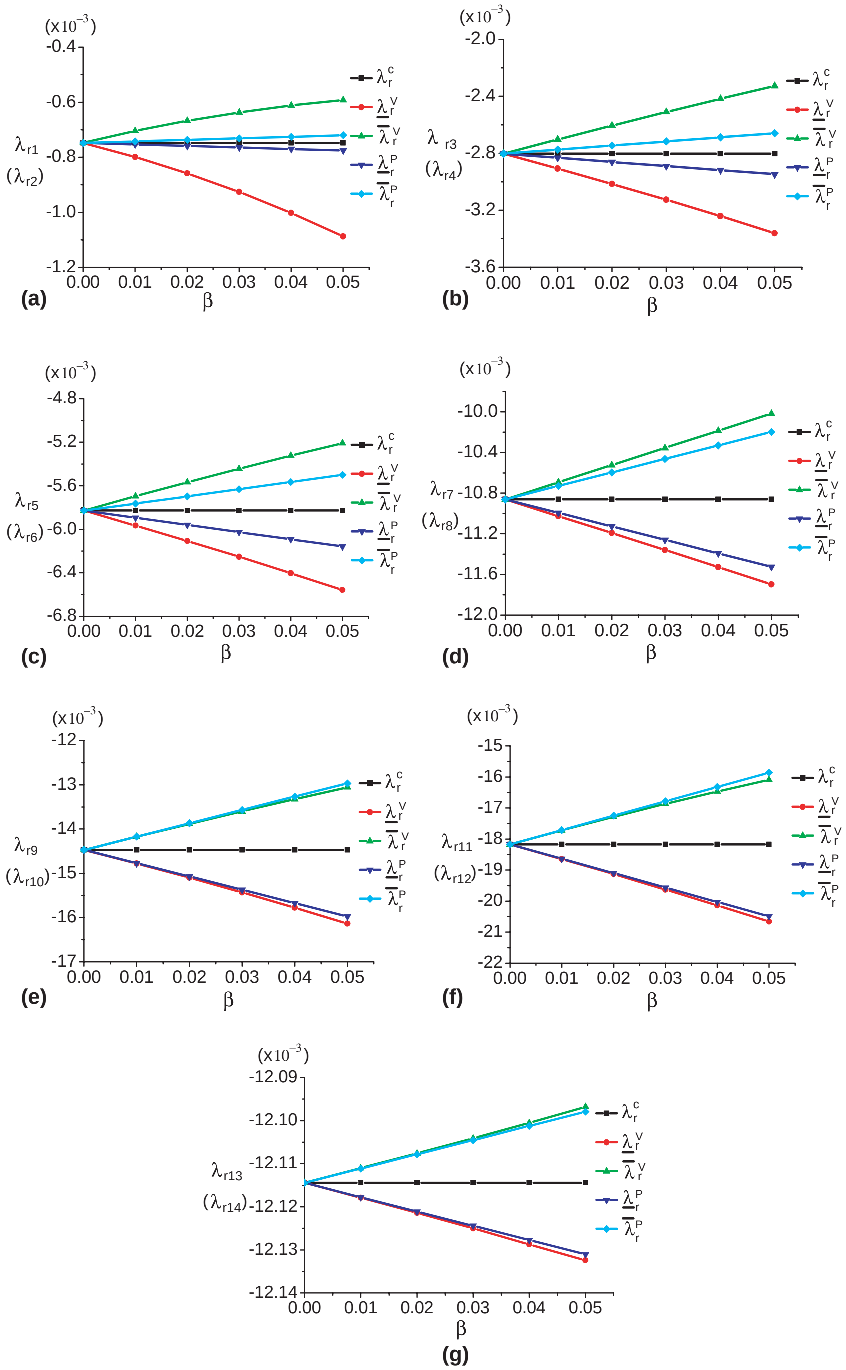

Fig. 2. Comparison of the region curves of the real part of complex eigenvalues yielded by the vertex solution theorem and the interval perturbation method.

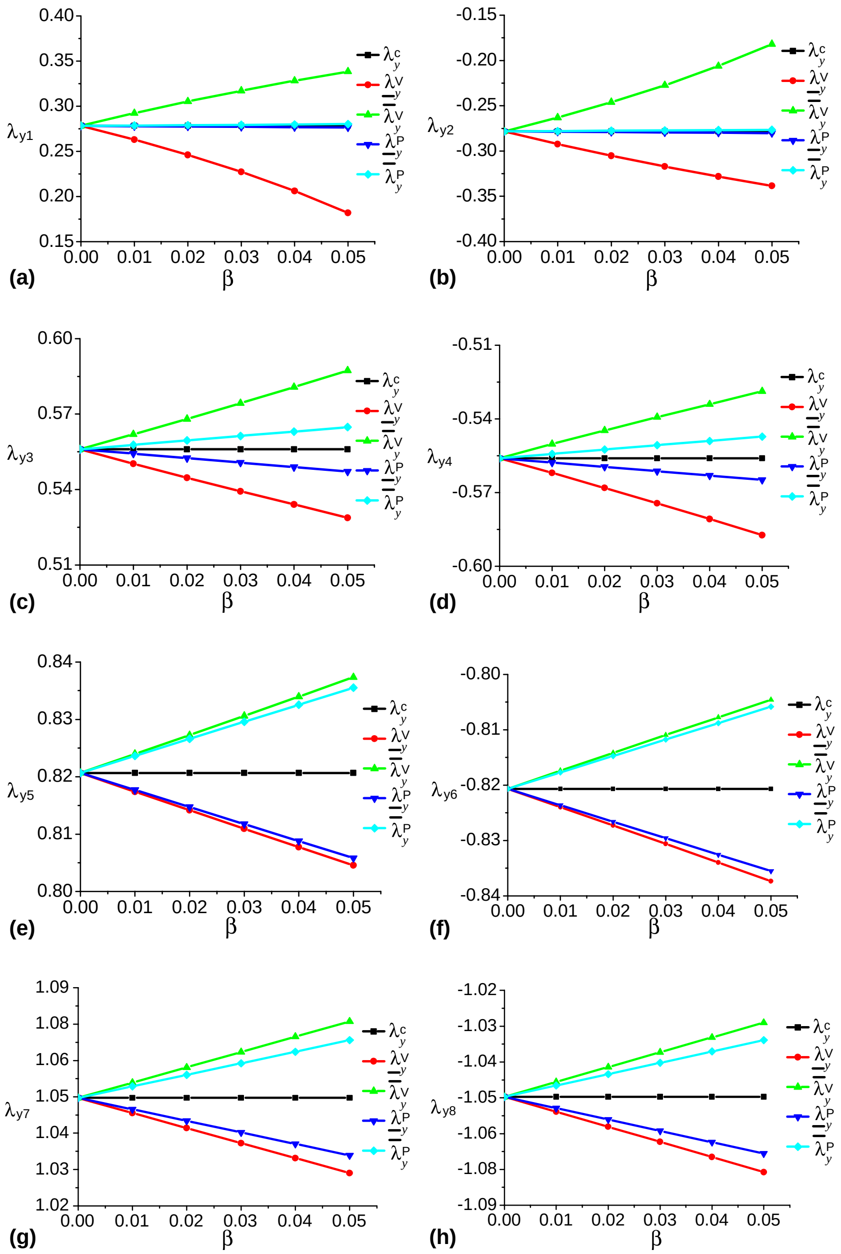

Fig. 3. Comparison of the region curves of the imaginary part of complex eigenvalues yielded by the vertex solution theorem and the interval perturbation method.

6. Conclusions

In this study, based on the interval mathematics and the optimization theory, an exact solution method—the vertex solution theorem and an interval perturbation method were presented for determining the range of complex eigenvalues.

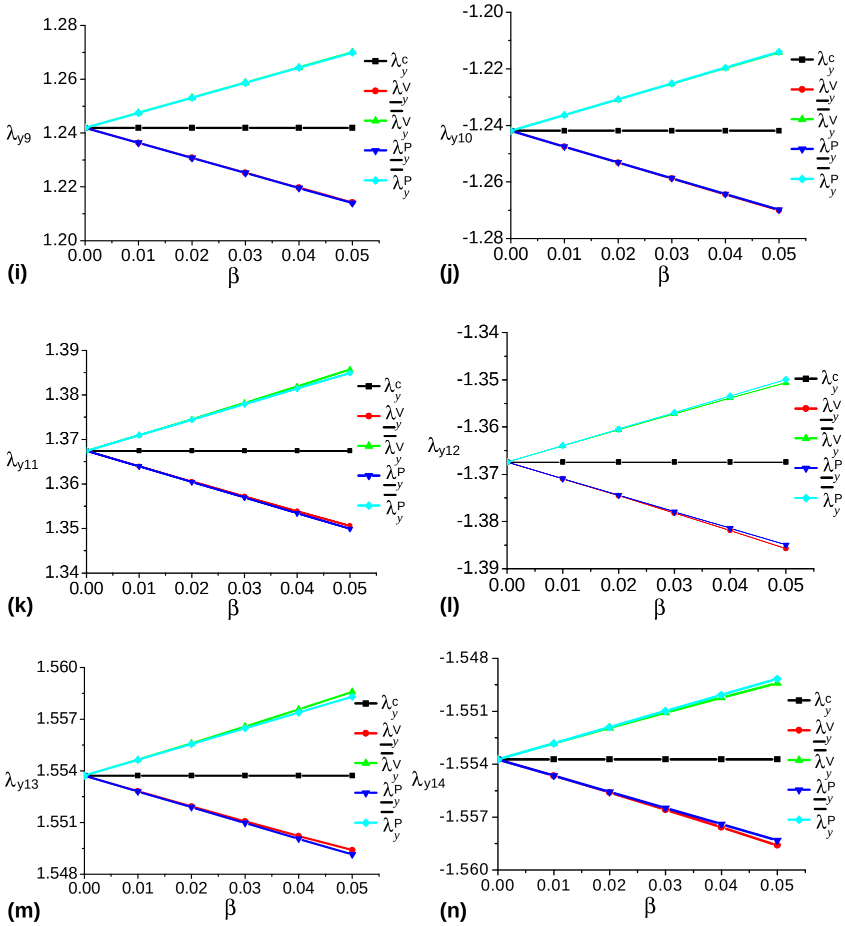

Fig. 3 (*continued*)

From the proof of the vertex solution theorem we can see that the computational efforts will grow sharply with the degree of uncertainty increasing. It shows the inefficiency of such approach for the solution of practical application. As we known that the vertex (exact) solution of the linear interval systems is NP-hard proved by Rohn and Kreinovich (1995), the vertex solution theorem for the generalized eigenvalue problem of interval matrices is also NP-hard. Nevertheless, the aim of this paper is mainly to provide a proof for the vertex solution theorem for the vibration frequencies of systems with uncertainty. Although such a formulation may not represent a significant contribution toward the advancement of engineering sciences, the vertex solution theorem can be used as a way to verify the sharpness of non NP-hard approximate solutions to the systems with interval coefficient matrices.

The generalized interval complex eigenvalue problem of interval matrices is by far more intricate than the generalized interval real eigenvalue problem of interval matrices. If one views the uncertainty of the interval matrix as a perturbation around the midpoint of the interval matrix, one can solve the generalized interval eigenvalue problem by the perturbation method. By applying the interval extension to the matrix perturbation formulation, we present the interval perturbation approximating formula for estimating the upper and lower bounds on the set of all possible eigenvalues of the generalized interval complex eigenvalue problem of the real symmetric non-positive definite interval matrix. Weak application condition and inexpensive computational efforts are mainly characteristics of the presented interval perturbation method.

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Appendix A. Minmax theorem for the generalized complex eigenvalues

Let $S_k, 1 \leqslant k \leqslant 2n$ denote an arbitrary k-dimensional subspace of complex space C^{2n}, and A and B be the $2n \times 2n$-dimensional real symmetric non-positive definite matrices with $2n$ complex eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{2n}$. Let the real parts and imaginary parts of the complex eigenvalues of the matrix pair $\langle A, B\rangle$ be ordered in increasing size, $\lambda_{r1} \leqslant \lambda_{r2} \leqslant \cdots \leqslant \lambda_{r2n}$ and $\lambda_{y1} \leqslant \lambda_{y2} \leqslant \cdots \leqslant \lambda_{y2n}$, and k be an integer with $1 \leqslant k \leqslant 2n$. Then, for the real part of complex eigenvalues

$$\lambda_{rk} = \min_{w_1, w_2, \ldots, w_{2n-k} \in C^{2n}} \max_{\substack{x \neq 0, x \in C^{2n} \\ x \perp w_1, w_2, \ldots, w_{2n-k}}} \mathrm{Re}\left\{\frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} B x}\right\}, \quad k = 1, 2, \ldots, 2n \tag{A.1}$$

and

$$\lambda_{rk} = \max_{w_1, w_2, \ldots, w_{k-1} \in C^{2n}} \min_{\substack{x \neq 0, x \in C^{2n} \\ x \perp w_1, w_2, \ldots, w_{k-1}}} \mathrm{Re}\left\{\frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} B x}\right\}, \quad k = 1, 2, \ldots, 2n \tag{A.2}$$

and for the imaginary part of complex eigenvalues

$$\lambda_{yk} = \min_{w_1, w_2, \ldots, w_{2n-k} \in C^{2n}} \max_{\substack{x \neq 0, x \in C^{2n} \\ x \perp w_1, w_2, \ldots, w_{2n-k}}} \mathrm{Im}\left\{\frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} B x}\right\}, \quad k = 1, 2, \ldots, 2n \tag{A.3}$$

and

$$\lambda_{yk} = \max_{w_1, w_2, \ldots, w_{k-1} \in C^{2n}} \min_{\substack{x \neq 0, x \in C^{2n} \\ x \perp w_1, w_2, \ldots, w_{k-1}}} \mathrm{Im}\left\{\frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} B x}\right\}, \quad k = 1, 2, \ldots, 2n \tag{A.4}$$

Proof. Only Eq. (A.1) is considered, and the proofs for Eqs. (A.2)–(A.4) are similar. Write $A = (X^T)^{-1} \Lambda X^{-1}$ and $B = (X^{-1})^{T}\ {\rm X}^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n})$, and let $1 \leqslant k \leqslant 2n$. If $x \neq 0$, then

$$\frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} B x} = \frac{(X^{-1}x)^{\mathrm{T}} \Lambda (X^{-1}x)}{x^{\mathrm{T}} B x} = \frac{(X^{-1}x)^{\mathrm{T}} \Lambda (X^{-1}x)}{(X^{-1}x)^{\mathrm{T}} (X^{-1}x)} \tag{A.5}$$

and $\{X^{-1}x : x \in C^{2n} \text{ and } x \neq 0\} = \{y \in C^{2n} : y \neq 0\}$. Thus, if $w_1, w_2, \ldots, w_{2n-k} \in C^{2n}$ are given, we have

$$\begin{aligned}
\max_{\substack{x\neq 0\\ x\perp w_1,w_2,\ldots,w_{2n-k}}} \mathrm{Re}\left\{\frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}Bx}\right\} &= \max_{\substack{y\neq 0\\ y\perp X^{-1}w_1,X^{-1}w_2,\ldots,X^{-1}w_{2n-k}}} \mathrm{Re}\left\{\frac{y^{\mathrm{T}}\Lambda y}{y^{\mathrm{T}}y}\right\} \\
&= \max_{\substack{y^{\mathrm{T}}y=1\\ y\perp X^{-1}w_1,X^{-1}w_2,\ldots,X^{-1}w_{2n-k}}} \mathrm{Re}\left\{\sum_{i=1}^{2n}\lambda_i|y_i|^2\right\} \\
&\geqslant \max_{\substack{y^{\mathrm{T}}y=1\\ y\perp X^{-1}w_1,X^{-1}w_2,\ldots,X^{-1}w_{2n-k}\\ y_1=y_2=\cdots=y_{2n-k}=0}} \mathrm{Re}\left\{\sum_{i=1}^{2n}\lambda_i|y_i|^2\right\} \\
&= \max_{\substack{|y_k|^2+|y_{k+1}|^2+\cdots+|y_{2n}|^2=1\\ y\perp X^{-1}w_1,X^{-1}w_2,\ldots,X^{-1}w_{2n-k}}} \mathrm{Re}\left\{\sum_{i=1}^{2n}\lambda_i|y_i|^2\right\} \geqslant \lambda_{rk}
\end{aligned} \tag{A.6}$$

This shows that

$$\max_{\substack{x\neq 0, x\in C^{2n}\\ x\perp w_1,w_2,\ldots,w_{2n-k}}} \mathrm{Re}\left\{\frac{xAx}{xBx}\right\} \geqslant \lambda_k \tag{A.7}$$

for any $2n-k$ vectors $w_1, w_2, \ldots, w_{2n-k}$. This inequality Eq. (A.7) becomes an equality if we choose $w_i = w_{2n-i+1}$. Therefore

$$\min_{w_1,w_2,\ldots,w_{2n-k}} \max_{\substack{x\neq 0\\ x\perp w_1,w_2,\ldots,w_{2n-k}}} \mathrm{Re}\left\{\frac{xAx}{xBx}\right\} = \lambda_{rk} \tag{A.8}$$

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